KOSHI TIPIDAGI SINGULYAR INTEGRALLAR UCHUN OPTIMAL KVADRATUR FORMULANI HISOBLASH ALGORITMI VA UNING YUPQA PROFIL NAZARIYASI MASALASIGA TATBIG‘I

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Annotatsiya. Maqolada Sobolevning $L_2^{(m)}(-1,1)$ fazosida Koshi tipidagi singulyar integrallarni taqribiy hisoblash uchun Sard ma’nosidagi optimal kvadratur formulani sonli amalga oshirish masalasi qaralgan. $m = 1$ holda optimal koeffitsiyentlar yopiq formula bilan, $m = 2$ holda uch diagonalli sistemaga keltiriladigan $O(N)$ murakkablikdagi algoritm bilan hisoblanadi. Sonli tajribalar xatolik funksionalining normasi $h^{m-1/2}$ tartibda, silliq funksiyalardagi haqiqiy xatolik esa $m = 1$ uchun h^2, $m = 2$ uchun taxminan h^3 tartibda kamayishini ko‘rsatdi. Algoritm yupqa simmetrik profil sirtidagi bosim koeffitsiyentini hisoblashga qo‘llangan.

Kalit so‘zlar: Koshi tipidagi singulyar integral, optimal kvadratur formula, Sobolev fazosi, xatolik funksionali, hisoblash algoritmi, matematik model, yupqa profil nazariyasi, bosim koeffitsiyenti.

Abstract. The paper deals with the numerical implementation of the Sard-optimal quadrature formula for Cauchy type singular integrals in the Sobolev space $L_2^{(m)}(-1,1)$. For $m = 1$ the coefficients are given in closed form; for $m = 2$ they are computed by an $O(N)$ algorithm that reduces to a tridiagonal system. Numerical experiments show that the norm of the error functional decays as $h^{m-1/2}$, while the actual error for smooth functions decays as h^2 for $m = 1$ and roughly as h^3 for $m = 2$. The algorithm is applied to computing the pressure coefficient on a thin symmetric airfoil.

Keywords: Cauchy type singular integral, optimal quadrature formula, Sobolev space, error functional, computational algorithm, mathematical model, thin airfoil theory, pressure coefficient.

Kirish. Aerodinamika, elastiklik nazariyasi va elektrodinamikaning ko‘plab matematik modellari Koshi yadroli singulyar integral tenglamalarga keltiriladi [1]. Ularni sonli yechishning asosiy bosqichi – Koshi bosh qiymati ma’nosidagi singulyar integralni taqribiy hisoblash. Integral oddiy ma’noda uzoqlashuvchi bo‘lgani uchun klassik kvadratur formulalarni bevosita qo‘llab bo‘lmaydi.

Adabiyotlar tahlili va metodologiya

Koshi tipidagi singulyar integrallarni hisoblash uchun diskret girdoblar usuli [2], Gauss tipidagi va interpolyatsion formulalar [3], splaynlarga asoslangan formulalar [4] hamda

diskret girdoblar usulining modifikatsiyalari [5] ishlab chiqilgan. Alohida yo‘nalish – Sard ma’nosidagi optimal formulalar: tugunlar tayinlangan holda xatolik funksionalining normasi koeffitsiyentlar bo‘yicha minimallashtiriladi [6]. S.L. Sobolev bu masalani differensial operatorning diskret analogi yordamida yechish usulini taklif etgan [7]. Koshi tipidagi integrallar uchun $L_2^{(m)}(-1,1)$ fazosida optimal koeffitsiyentlarning oshkor ko‘rinishi [8–10] ishlarda olingan. Ushbu ishning maqsadi – shu formulani hisoblash nuqtayi nazaridan tadqiq qilish: koeffitsiyentlarni tejamli hisoblash algoritmini berish, xatolikning tugunlar soniga bog‘liqligini sonli o‘rganish hamda algoritmni yupqa profil nazariyasining matematik modeliga qo‘llash.

Quyidagi kvadratur formulani qaraymiz:

$$\text{p.v.} \int_{-1}^{1} \frac{\varphi(x)}{x-t}\,dx \cong \sum_{\beta=0}^{N} C_\beta\, \varphi(x_\beta), \qquad -1 < t < 1, \qquad (1)$$

bu yerda C_β – koeffitsiyentlar, $x_\beta = h\beta - 1$ – tugunlar, $h = 2/N$, φ – $L_2^{(m)}(-1,1)$ fazosining elementi. Bu fazoda yarim norma sifatida $\varphi^{(m)}$ hosilaning $L_2(-1,1)$ dagi normasi olinadi. (1) formulaning xatoligi ℓ xatolik funksionalining φ dagi qiymatiga teng bo‘lib, Koshi–Shvars tengsizligiga ko‘ra $|(\ell,\varphi)| \le \|\ell\|\cdot\|\varphi\|$. Sard masalasi $\|\ell\|$ normani eng kichik qiladigan C_β koeffitsiyentlarni topishdan iborat; bunda formula darajasi $m-1$ dan oshmaydigan ko‘phadlarda aniq bo‘lishi shart. Norma kvadrati quyidagicha ifodalanadi [7, 8]:

$$\|\ell\|^2 = (-1)^m \left[\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta C_\gamma G_m(x_\beta - x_\gamma) - 2\sum_{\beta=0}^{N} C_\beta\, f_m(x_\beta) + J_m(t)\right], \qquad (2)$$

bu yerda

$$G_m(x) = \frac{|x|^{2m-1}}{2(2m-1)!}, \quad f_m(y) = \text{p.v.} \int_{-1}^{1} \frac{G_m(x-y)}{x-t}\,dx, \quad J_m(t) = \text{p.v.} \int_{-1}^{1} \frac{f_m(y)}{y-t}\,dy.$$

Hisoblashlar Python tilida mpmath kutubxonasi yordamida 30–50 xonali arifmetikada bajarildi.

Natijalar

$F(y) = (y-t)\ln|y-t|$, $F(t) = 0$ deb belgilaymiz.

1-teorema. $L_2^{(1)}(-1,1)$ fazosida (1) ko‘rinishdagi optimal kvadratur formulaning koeffitsiyentlari quyidagicha:

$$C_0 = \frac{F(x_1) - F(x_0)}{h} - 1 - \ln(1+t), \quad C_N = \frac{F(x_{N-1}) - F(x_N)}{h} + 1 + \ln(1-t),$$

$$C_\beta = \frac{F(x_{\beta-1}) - 2F(x_\beta) + F(x_{\beta+1})}{h}, \qquad \beta = 1,2,\ldots,N-1. \qquad (3)$$

Isbot quyidagi tamoyilga tayanadi: Gilbert fazosida chegaralangan funksionalni tugun qiymatlari bo‘yicha eng yaxshi yaqinlashtirish uni interpolyatsion splaynda aniq hisoblashga teng kuchli [11]. $m = 1$ da bu splayn bo‘lakli-chiziqli bo‘lib, uni hadma-had integrallash (3) ni beradi. (3) dan ko‘rinadiki, t ichki tugun bilan ustma-ust tushsa, $t = x_k$, u holda $C_k = 0$.

$m = 2$da interpolyatsion splayn – natural kubik splayn. Quyidagi yordamchi funksiya va vaznlarni kiritamiz ($\sigma = 0$ va $\sigma = 1$ da logarifmli had nolga teng deb olinadi):

$$K(\sigma) = -\frac{2}{3} + \frac{\sigma}{2} + \sigma^2 + \sigma(\sigma^2 - 1)\ln\left|\frac{1-\sigma}{\sigma}\right|,$$

$$b_i = \frac{h^2}{6}\left[K\left(\frac{t - x_{i-1}}{h}\right) - K\left(\frac{x_{i+1} - t}{h}\right)\right], \qquad i = 1, \ldots, N-1. \qquad (4)$$

Algoritm. 1) (3) formula bo‘yicha a_β, $\beta = 0, \ldots, N$ qiymatlar hisoblanadi; 2) (4) formula bo‘yicha b_i, $i = 1, \ldots, N-1$ qiymatlar hisoblanadi; 3) $y_{i-1} + 4y_i + y_{i+1} = b_i$, $i = 1, \ldots, N-1$, $y_0 = y_N = 0$ uch diagonalli sistema haydash usuli bilan yechiladi; 4) koeffitsiyentlar yig‘iladi ($y_{-1} = y_{N+1} = 0$):

$$C_\beta = a_\beta + \frac{6}{h^2}\left(y_{\beta-1} - 2y_\beta + y_{\beta+1}\right), \qquad \beta = 0,1, \ldots, N. \qquad (5)$$

2-teorema. Algoritm natijasi $L_2^{(2)}(-1,1)$ fazosidagi (1) ko‘rinishdagi optimal kvadratur formulaning koeffitsiyentlaridir.

Algoritm $O(N)$ amal talab qiladi, optimallik shartlaridan kelib chiqadigan to‘la matritsali chiziqli sistemani to‘g‘ridan-to‘g‘ri yechish esa $O(N^3)$ amalni talab etadi. Uch diagonalli sistema matritsasi qat’iy diagonal ustunlikka ega bo‘lgani uchun haydash usuli turg‘un. $N = 10$ va $N = 20$ da 50 xonali arifmetikada ikkala usul bilan topilgan koeffitsiyentlar farqi 10^{-46} dan oshmadi. $m = 1$ dan farqli ravishda, $m = 2$ da t tugunga tushganda shu tugundagi koeffitsiyent umuman olganda nolga teng emas.

Amaliy masala. Yupqa profil nazariyasida [12] nol hujum burchagi ostida aylanib oqilayotgan simmetrik profil sirtidagi tezlik g‘alayoni u va bosim koeffitsiyenti C_p profilning yarim qalinligi $\eta(x)$ orqali singulyar integral bilan ifodalanadi:

$$\frac{u(t)}{V_\infty} = -\frac{1}{\pi}\,\text{p.v.}\int_{-1}^{1}\frac{\eta'(x)}{x-t}\,dx, \qquad C_p(t) = -\frac{2u(t)}{V_\infty}, \qquad (6)$$

bu yerda V_∞ – oqim tezligi, profil vatari $[-1,1]$ kesma. (6) dagi integral (1) ko‘rinishda bo‘lib, $\varphi = \eta'$. Model sifatida nisbiy qalinligi $\tau = 0,1$ bo‘lgan $\eta(x) = \tau\cos(\pi x/2)$ profil olindi; uning uchun aniq yechim integral sinus va integral kosinus orqali ifodalanadi. 1-jadvalda $t = 0$ nuqta uchun (2) bo‘yicha hisoblangan norma hamda C_p absolyut xatoligining t bo‘yicha $[-0,9;\ 0,9]$ kesmadagi eng katta qiymati E keltirilgan (qadam 0,01).

1-jadval. Xatolik funksionalining normasi ($t = 0$) va C_p ning eng katta xatoligi

N	$\|\ell\|$, $m = 1$	$\|\ell\|$, $m = 2$	E, $m = 1$	E, $m = 2$
20	0,46303	$9,9418 \cdot 10^{-3}$	$6,6382 \cdot 10^{-4}$	$2,5474 \cdot 10^{-4}$
40	0,32805	$3,5169 \cdot 10^{-3}$	$1,6566 \cdot 10^{-4}$	$9,5656 \cdot 10^{-6}$
80	0,23219	$1,2438 \cdot 10^{-3}$	$4,1331 \cdot 10^{-5}$	$9,1960 \cdot 10^{-7}$
160	0,16426	$4,3982 \cdot 10^{-4}$	$1,0331 \cdot 10^{-5}$	$1,2492 \cdot 10^{-7}$

Nikki baravar ortganda norma $m = 1$ da $2^{0,5}$, $m = 2$ da $2^{1,5}$ marta kamayadi, ya’ni $\|\ell\| = O\left(h^{m-1/2}\right)$. Haqiqiy xatolikning kuzatilgan tartibi $m = 1$ uchun 2,0, $m = 2$ uchun $N \geq 80$ da 2,9–3,4. Profil o‘rtasida bosim eng past: $C_p(0) = -0,27415$; $N = 40$ da bu qiymat $m = 1$ formula bilan $1,4 \cdot 10^{-4}$, $m = 2$ formula bilan $1,5 \cdot 10^{-6}$ xatolikda topiladi.

Muhokama

Olingan natijalar to‘rtta xulosaga olib keladi. Birinchidan, norma $h^{m-1/2}$ tartibda kamayadi; bu regulyar integrallar uchun optimal formulalarga xos $O(h^m)$ tartibdan yarim

birlikka past bo‘lib, yo‘qotish yadroning maxsusligi bilan bog‘liq. Ikkinchidan, Koshi–Shvars bahosi kafolatlangan, lekin silliq funksiyalar uchun qo‘pol: amaliy masalada $t = 0$, $N = 40$, $m = 2$ uchun u $8{,}7 \cdot 10^{-4}$ ni beradi, bu haqiqiy xatolikdan qariyb 570 marta katta. Uchinchidan, $m = 2$ formulaning aniqligi kesma chetlari yaqinida pasayadi: natural splaynning chegaradagi $S''(\pm 1) = 0$ sharti qo‘shimcha xatolik kiritadi va u ichkariga qarab taxminan 0,27 maxrajli geometrik progressiya kabi so‘nadi. To‘rtinchidan, $m = 2$ formula $N = 40$ da $9{,}6 \cdot 10^{-6}$ aniqlikni beradi, $m = 1$ formula esa shunga yaqin aniqlikka $(1{,}0 \cdot 10^{-5})$ faqat $N = 160$ da erishadi, ya’ni model bo‘yicha hisoblash hajmi to‘rt marta qisqaradi.

Xulosa

$m = 1$ hol uchun optimal koeffitsiyentlar bitta funksiyaning chekli ayirmalari ko‘rinishida yozildi, $m = 2$ hol uchun $O(N)$ murakkablikdagi algoritm taklif etildi va to‘g‘ridan-to‘g‘ri yechim bilan tekshirildi. Sonli tajribalar xatolik funksionali normasining kamayish tartibini aniqlash imkonini berdi. Algoritm yupqa simmetrik profil sirtidagi bosim koeffitsiyentini $N = 40$ da 10^{-5} dan kichik xatolik bilan hisobladi.

Taklif va tavsiyalar

1. Sonli kuzatilgan $\| \ell \| = O\left(h^{m-1/2}\right)$ bahoni ixtiyoriy m uchun qat’iy isbotlash va uning asimptotik doimiysini topish taklif etiladi.

2. Singulyar nuqta oldindan ma’lum bo‘lgan masalalarda tugunlar to‘rini t qismiy oraliq o‘rtasiga tushadigan qilib tanlash tavsiya etiladi: hisoblashlarga ko‘ra ($N = 100$) bunda norma t tugunda bo‘lgan holga qaraganda $m = 2$ uchun 2,6 marta kichik.

3. Kesma chetlari yaqinidagi aniqlikni oshirish uchun chegaraviy tugunlarda funksiya hosilalari qiymatlaridan foydalanadigan optimal formulalarni [9] tadqiq qilish maqsadga muvofiq.

4. Ko‘taruvchi kuch masalasidagi singulyar integral tenglamaning yechimi profil chetlarida maxsuslikka ega; uni yechish uchun taklif etilgan algoritmni vaznli optimal formulalarga umumlashtirish tavsiya etiladi.

Foydalanilgan adabiyotlar

1. Гахов Ф.Д. Краевые задачи. – М.: Наука, 1977. – 640 с.
2. Белоцерковский С.М., Лифанов И.К. Численные методы в сингулярных интегральных уравнениях. – М.: Наука, 1985. – 256 с.
3. Monegato G. The numerical evaluation of one-dimensional Cauchy principal value integrals // Computing. – 1982. – Vol. 29. – P. 337–354.
4. Dagnino C., Santi E. Spline product quadrature rules for Cauchy singular integrals // Journal of Computational and Applied Mathematics. – 1990. – Vol. 33. – P. 133–140.
5. Eshkuvatov Z.K., Nik Long N.M.A., Abdulkawi M. Numerical evaluation for Cauchy type singular integrals on the interval // Journal of Computational and Applied Mathematics. – 2010. – Vol. 233, No. 8. – P. 1995–2001.
6. Sard A. Best approximate integration formulas; best approximation formulas // American Journal of Mathematics. – 1949. – Vol. 71. – P. 80–91.
7. Соболев С.Л. Введение в теорию кубатурных формул. – М.: Наука, 1974. – 808 с.

8. Shadimetov Kh.M., Hayotov A.R., Akhmedov D.M. Optimal quadrature formulas for Cauchy type singular integrals in Sobolev space // Applied Mathematics and Computation. – 2015. – Vol. 263. – P. 302–314.
9. Akhmedov D.M., Hayotov A.R., Shadimetov Kh.M. Optimal quadrature formulas with derivatives for Cauchy type singular integrals // Applied Mathematics and Computation. – 2018. – Vol. 317. – P. 150–159.
10. Akhmedov D.M. Construction of optimal quadrature formulas for Cauchy type singular integrals in the $L_2^{(m)}(-1,1)$ space // Uzbek Mathematical Journal. – 2025. – Vol. 69, No. 3. – P. 15–24.
11. Schoenberg I.J. Spline interpolation and best quadrature formulae // Bulletin of the American Mathematical Society. – 1964. – Vol. 70. – P. 143–148.
12. Katz J., Plotkin A. Low-Speed Aerodynamics. – 2nd ed. – Cambridge: Cambridge University Press, 2001. – 613 p.