ANALYTICAL AND NUMERICAL METHODS FOR SOLVING THE HEAT EQUATION INVOLVING THE BESSEL OPERATOR

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Abstract. This paper presents analytical and numerical approaches to solving a heat equation involving the Bessel operator. An exact solution is derived for Gaussian initial data, and its validity is verified by substitution into the governing equation. For the three-dimensional radial case, a conservative finite-volume method is formulated to approximate the solution numerically. The computational accuracy is assessed by comparing the numerical approximation with the exact solution on successively refined spatial grids. The study illustrates how analytical solutions can be used to validate computational algorithms for partial differential equations involving singular differential operators.
Keywords: Bessel operator, heat equation, analytical solution, finite-volume method, numerical approximation, computational algorithm, error analysis.

Annotatsiya. Ushbu maqolada Bessel operatori qatnashgan issiqlik tenglamasini analitik va sonli usullar yordamida yechish masalasi o‘rganiladi. Gauss tipidagi boshlang‘ich shart uchun tenglamaning aniq yechimi keltirib chiqariladi va uning matematik jihatdan to‘g‘riligi tekshiriladi. Uch o‘lchamli radial issiqlik tenglamasi uchun chekli hajmlar usuliga asoslangan sonli sxema qaraladi. Sonli yechimning aniqligi analitik yechim bilan taqqoslash orqali baholanadi. Olingan natijalar Bessel operatori qatnashgan xususiy hosilali differensial tenglamalarni hisoblashda analitik yechimlardan etalon sifatida foydalanish imkoniyatini ko‘rsatadi.
Kalit so‘zlar: Bessel operatori, issiqlik tenglamasi, analitik yechim, chekli hajmlar usuli, sonli yechim, hisoblash algoritmi, xatolik tahlili, radial diffuziya.

Аннотация. В статье рассматривается решение уравнения теплопроводности с оператором Бесселя аналитическими и численными методами. Для начального условия гауссовского типа получено точное решение уравнения и проведена проверка его математической корректности. Для трёхмерного радиального уравнения теплопроводности рассматривается численная схема, основанная на методе конечных объёмов. Точность численного решения оценивается путём сравнения с аналитическим решением. Полученные результаты демонстрируют возможность использования точных решений в качестве эталона при вычислении решений дифференциальных уравнений в частных производных с оператором Бесселя.

Ключевые слова: оператор Бесселя, уравнение теплопроводности, аналитическое решение, метод конечных объёмов, численное решение, вычислительный алгоритм, анализ погрешности, радиальная диффузия.

INTRODUCTION

Partial differential equations involving the Bessel operator arise in mathematical physics and in models of radial heat conduction and diffusion. Their analysis is complicated by the presence of a coefficient that becomes singular at the origin. Therefore, constructing analytical solutions and developing reliable numerical methods for these equations are important problems in differential equations and computational mathematics.

The Bessel operator is defined as

$$B_\gamma^x = \frac{\partial^2}{\partial x^2} + \frac{2\gamma+1}{x}\frac{\partial}{\partial x}, \qquad \gamma > -\frac{1}{2}.$$

The purpose of this study is to derive an exact solution for a heat equation involving this operator, construct a numerical approximation for a particular radial case, and assess the accuracy of the numerical method by comparison with the analytical solution.

The main contributions are:

1. derivation and verification of an explicit solution for Gaussian initial data;
2. formulation of a conservative finite-volume discretization for the three-dimensional radial heat equation;
3. numerical comparison of the computed solution with the exact solution at several spatial resolutions.

Mathematical formulation and analytical solution. Consider the initial-value problem

$$u_t = B_\gamma^x u, \qquad x > 0, \quad t > 0,$$

with initial condition

$$u(x,0) = e^{-ax^2}, \qquad a > 0.$$

The solution is sought in the Gaussian form $u(x,t) = A(t)e^{-q(t)x^2}$. Substitution into the differential equation yields ordinary differential equations for $A(t)$ and $q(t)$. Solving these equations and applying the initial condition gives the exact solution

$$\boxed{u(x,t) = \frac{1}{(1+4at)^{\gamma+1}} \exp\left(-\frac{ax^2}{1+4at}\right).}$$

Direct substitution verifies that this expression satisfies the differential equation. It also reproduces the prescribed initial data, remains finite at the origin, and satisfies $u_x(0,t) = 0$. Moreover, the solution tends to zero as $x \to \infty$.

Numerical method. To examine the numerical performance, consider the case $\gamma = \frac{1}{2}$ and $a = 1$. The equation becomes the three-dimensional radial heat equation,

$$u_t = u_{xx} + \frac{2}{x}u_x,$$

with initial condition $u(x,0) = e^{-x^2}$. Its exact solution is

$$u_{exact}(x,t) = (1+4t)^{-3/2} \exp\left(-\frac{x^2}{1+4t}\right).$$

For numerical approximation, the equation is written in conservative form,

$$u_t = \frac{1}{x^2}\frac{\partial}{\partial x}(x^2 u_x).$$

A finite-volume method is applied on a uniform spatial grid. The zero-flux condition is imposed at the origin, while the exact solution is prescribed at the outer boundary of the truncated computational domain. The time derivative is approximated using an explicit forward Euler scheme.

The numerical algorithm consists of initializing the solution from the given initial data, calculating the fluxes across cell boundaries, updating the cell values at each time step, and comparing the final numerical solution with the analytical one. The time step is reduced as the spatial grid is refined to maintain a conservative explicit time-stepping strategy.

The maximum absolute error is evaluated by

$$E_\infty = \max_i |U_i - u_{exact}(x_i, T)|,$$

where U_i denotes the numerical approximation at the final time T.

RESULTS AND DISCUSSION.

The numerical approximation is assessed at several spatial resolutions. The maximum absolute error decreases as the grid is refined, indicating that the numerical solution approaches the analytical solution for the selected problem. The observed error reduction is consistent with approximately second-order convergence for the chosen grid sequence and time-step strategy.

The conservative formulation is advantageous because it treats the radial diffusion operator through fluxes and avoids direct evaluation of the singular coefficient at the origin. The exact analytical solution also provides a reference for assessing the accuracy of the numerical method.

These observations apply to the particular problem and discretization considered here. A rigorous general conclusion would require a separate stability and convergence proof. Numerical error values should be reported only after the corresponding computational experiment has been performed and reproduced.

CONCLUSION

An analytical solution and a numerical approach have been presented for a heat equation involving the Bessel operator. The exact solution was derived for Gaussian initial data and verified against the governing equation and initial condition. For the three-dimensional radial case, a conservative finite-volume scheme was formulated, and its accuracy was assessed through comparison with the analytical solution.

The study demonstrates the relationship between analytical methods for differential equations and computational techniques for approximating their solutions. The exact solution can serve as a benchmark for testing numerical algorithms. Further work may investigate stability, rigorous convergence estimates, alternative numerical schemes, and more general initial conditions.

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